On the expansion of a gas into vacuum

By H. P. GREENSPAN AND D. S. BUTLER†

Mathematics Department, Massachusetts Institute of Technology

(Received 24 November 1961)

A study is made of the flow into vacuum of a gas initially at rest in a state of uniform pressure and density; the analysis is based on a continuum model. Among the topics discussed are the motion of the gas-vacuum interface, the reflexion of a plane front off a rigid wall, the propagation of compressive waves within such expansions, the escape from a sphere and the collapse of a spherical cavity.

1. Introduction

The basic problem studied here concerns the isentropic flow into vacuum of a gas whose initial state is one of constant pressure, P_0, constant density, ρ_0, and zero velocity.

The simple exact solution of the strictly one-dimensional problem for which the gas initially occupies an entire half space (see Stanyukovich 1960) is fundamental to all that follows and it is desirable to discuss it anew at this time, emphasizing those features we wish to develop and exploit.

Let l, $c_0 = (\gamma P_0/\rho_0)^{\frac{1}{2}}$, l/c_0, ρ_0, and $\rho_0 c_0^2$ represent respectively, the characteristic length, velocity, time, density and pressure. The equations of motion, in dimensionless form, are then

$$u_t + uu_x + \frac{2}{\gamma-1} cc_x = 0, \tag{1.1}$$

$$\frac{2}{\gamma-1}(c_t + uc_x) + cu_x = 0, \tag{1.2}$$

where u is the particle velocity, c is the local sound speed and γ is the ratio of specific heats. Here we have used the isentropic nature of the flow, i.e. $c^2 = \rho^{\gamma-1}$, and the fact that the gas is initially at rest in a uniform state to eliminate the pressure and density in favour of the local sound speed c.

Subsequent to release, the only particles in motion lie between the gas-vacuum interface and the rarefaction front propagating into the stationary gas. The interface is the zero sound speed surface $c = 0$, along which pressure and density are both zero; the velocity of this front is an unknown. The acoustic front, however, propagates with unit dimensionless velocity into the quiescent gas. Consequently, equations (1.1) and (1.2) must be solved subject to the condition $u = 0$ on $x = -t$ (the gas is taken to occupy the region $x < 0$). The solution may be obtained by either the method of characteristics or by the

† Permanent address: Armament Research and Development Establishment, Fort Halstead, Sevenoaks, Kent, England.

introduction of a similarity variable by which means the system is reduced to a set of ordinary differential equations. The latter method is used in § 3; application of the former procedure yields the results:

$$u+\frac{2}{\gamma-1}c = \text{const.} \quad \text{along } C_+ \text{ characteristics given by } dx/dt = u+c;$$

$$u-\frac{2}{\gamma-1}c = \text{const.} \quad \text{along } C_- \text{ characteristics given by } dx/dt = u-c.$$

The complete solution satisfying the boundary conditions is

$$u = \frac{2}{\gamma+1}(1+\eta), \tag{1.3}$$

$$c = \frac{\gamma-1}{\gamma+1}\left(\frac{2}{\gamma-1}-\eta\right), \tag{1.4}$$

with $\eta = x/t$, and for $-1 \leqslant \eta \leqslant 2/(\gamma-1)$. For values of $\eta \leqslant -1$, $u = 0$ and $c = 1$, whereas the domain $\eta \geqslant 2/(\gamma-1)$ corresponds to the vacuum. The C_+ family of characteristics are the curves

$$x = kt^{(3-\gamma)/(\gamma+1)} + \frac{2}{\gamma-1}t \tag{1.5}$$

(k is an arbitrary constant), and the C_- characteristics are the straight lines

$$x = \eta t; \tag{1.6}$$

the configuration is shown in figure 2.

Several properties of the solution are especially noteworthy. First, the gas-vacuum interface, the curve $c = 0$, is a straight line implying that the front accelerates instantaneously and thereafter moves with constant escape velocity. Secondly, on the interface $dx/dt = u$, so that this curve is also a particle path. It follows directly that the kinetic energy density is a maximum at the front whereas the potential energy density is a minimum there. (The particles constituting the front achieve a velocity larger than that possible in purely steady flows.) The redistribution of energy density over the mass of moving gas is one of the principal features of the unsteady escape process. Thirdly, the interface is genuinely a member of both families of characteristics curves, C_+ and C_-. It is not an envelope; in other words, no sound wave from the interior ever reaches the front. Fourthly, all derivatives are everywhere finite at non-zero times and, in particular, the following relationship holds

$$\lim_{c\to 0} cc_x = 0.$$

Stanyukovich (1960) considers several variations and generalizations of this basic problem. Among the topics discussed are the escape of a column of gas from both ends into vacuum, the escape of a gas from a tube of finite length, the expansion of a gaseous sphere into vacuum and the effects of body forces on such flows. However, there remain many interesting aspects of the problem which have not been dealt with as yet. For example, it is of interest to examine the behaviour of the gas-vacuum interface when the original gas container is arbitrary, and to inquire into the circumstances for which it may be useful to relate the interface with an infinitely strong shock wave. The present research is intended to shed some light on questions such as these.

Other authors, Copson (1950), Pack (1953) and Keller (1957), have considered related problems generally assuming different initial conditions. The last mentioned presents an entire class of similarity solutions for gaseous expansions from a state of rest into vacuum, but the initial pressure, density and sound

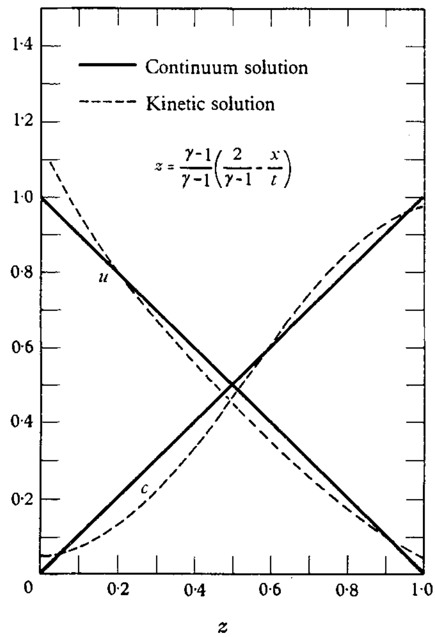

FIGURE 1. Comparison of continuum and collisionless-gas solutions.

speed in these cases are not uniform and constant. As a consequence, the gas-vacuum interface does not maintain a uniform velocity but is found to accelerate. However, we had best postpone a more complete discussion of the role played by the initial conditions for the time being.

Keller (1948) has compared the foregoing solution to that obtained by solving the Boltzmann equation in the case of the expansion of a collision-free gas into vacuum. The results shown in figure 1 indicate surprisingly good agreement between the two approaches. Evidently in such problems, results derivable from a continuum model apply at much lower density levels than might be expected.

The gas-vacuum interface is related, in a sense, to an infinitely strong shock wave. If one gas is allowed to escape into another rest gas, the resultant flow may be divided into three regions, including a shock proceeding into the rest gas, a stationary rarefaction wave and a simple rarefaction wave into the escaping gas. If, furthermore, the density of the rest gas tends to zero, the entire shock régime including the stationary wave become meaningless inasmuch as the density also vanishes in this domain. The shock, however, becomes infinitely strong, i.e. the density ratio approaches $(\gamma+1)/(\gamma-1)$, even though the density is itself zero. The only sensible motion then occurs behind the simple wave (gas-vacuum interface) which must by itself represent the degeneracy of the entire system. In other words we would expect the interface to behave in certain respects as an infinitely strong shock. In order to illustrate this feature, we treat now the reflexion of a plane front off a wall showing that the front does indeed reflect as an infinitely strong shock wave. Stanyukovich considered this problem, but the concise treatment given here is different and many of the results are new.

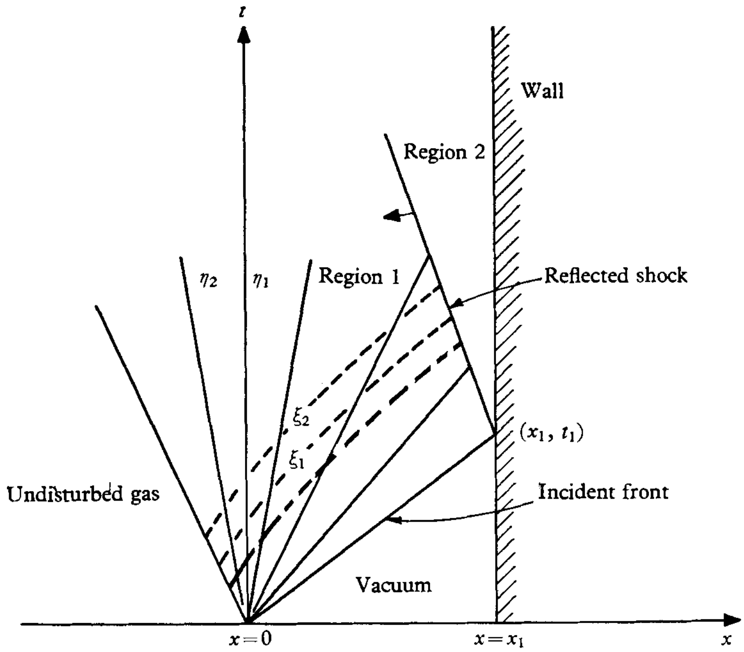

FIGURE 2. Reflexion off a rigid wall—configuration.

Suppose that the plane front impinges on a wall at $x = x_1$. It will arrive at time $t_1 = \frac{1}{2}(\gamma-1)x_1$. The gas particles in the front are moving with velocity $2/(\gamma-1)$ as they approach the wall and must come to rest immediately after impact. Therefore, the flow variables are multivalued at (x_1, t_1) and a reflected shock wave must be generated in order to accomplish the required deceleration. Furthermore, the similarity variable $\xi = (x-x_1)/(t-t_1)$ is appropriate near (x_1, t_1), since it is consistent with the uniform motion of the particles in the front.

The flow in region 1, ahead of the reflected shock (see figure 2), is independent of the presence of the wall and the particle velocity and sound speed are given by the expressions (1.3) and (1.4). These are expanded in powers of $(t-t_1)$ with coefficients depending on ξ:

$$u = \frac{2}{(\gamma-1)} + \frac{2}{(\gamma+1)} \frac{(\xi-\xi_0)}{t_1}(t-t_1) + \ldots, \tag{1.7}$$

$$c = \frac{(\gamma-1)}{(\gamma+1)} \frac{(\xi-\xi_0)}{t_1}(t-t_1) + \ldots, \tag{1.8}$$

where $\xi = 2/(\gamma-1) = \xi_0$ is the incident front. Since the flow is homentropic, the corresponding expansions for the density and pressure are

$$\rho = c^{2/(\gamma-1)} = (t-t_1)^{2/(\gamma-1)}\left[\left\{\frac{(\gamma-1)(\xi-\xi_0)}{(\gamma+1)t_1}\right\}^{2/(\gamma-1)}+\dots\right], \tag{1.9}$$

$$p = \gamma^{-1}c^{2\gamma/(\gamma-1)} = (t-t_1)^{2\gamma/(\gamma-1)}\left[\frac{1}{\gamma}\left\{\frac{(\gamma-1)(\xi-\xi_0)}{(\gamma+1)t_1}\right\}^{2\gamma/(\gamma-1)}+\dots\right]. \tag{1.10}$$

The flow in region 2, behind the shock, is non-homentropic and governed by the equations

$$\rho_t+u\rho_x+\rho u_x = 0, \tag{1.11}$$

$$\rho(u_t+uu_x)+p_x = 0, \tag{1.12}$$

$$p_t+up_x-\frac{\gamma p}{\rho}(\rho_t+u\rho_x) = 0. \tag{1.13}$$

The solution must satisfy the boundary condition $u = 0$ on $x = x_1$ and must be matched with the known solution for region 1, across a shock discontinuity.

For small $(t-t_1)$, the total change in particle velocity across the shock and region 2 is $2/(\gamma-1) = O(1)$. If an appreciable proportion of this change occurs across the shock front, its velocity must also be $O(1)$. The sound speed, density and pressure of the gas ahead of the shock wave are $O(t-t_1)$, $O(t-t_1)^{2/(\gamma-1)}$ and $O(t-t_1)^{2\gamma/(\gamma-1)}$, respectively (see equations (1.8) to (1.10)). Therefore, the strong-shock approximation to the Rankine–Hugoniot conditions is valid for small $(t-t_1)$ and the density, pressure and particle velocity immediately behind the shock are $O(t-t_1)^{2/(\gamma-1)}$, $O(t-t_1)^{2/(\gamma-1)}$ and $O(1)$, respectively. The appropriate form of the solution in region 2 is therefore

$$\rho = (t-t_1)^{2/(\gamma-1)}[\rho_0(\xi)+(t-t_1)\rho_1(\xi)+\dots], \tag{1.14}$$

$$p = (t-t_1)^{2/(\gamma-1)}[p_0(\xi)+(t-t_1)p_1(\xi)+\dots], \tag{1.15}$$

$$u = u_0(\xi)+(t-t_1)u_1(\xi)+\dots, \tag{1.16}$$

and the position and velocity of the shock front are

$$\xi = U_0+\tfrac{1}{2}U_1(t-t_1)+\dots, \tag{1.17}$$

and

$$U = U_0+U_1(t-t_1)+\dots, \tag{1.18}$$

where U_0, U_1, etc., are constants. The boundary condition at the wall implies that

$$u_0(0) = u_1(0) = \dots = 0.$$

When the expressions (1.14) to (1.16) are substituted into equations (1.11) to (1.13), and the lowest-order terms are equated, the following equations for ρ_0, p_0 and u_0 are obtained:

$$2\rho_0/(\gamma-1)+(u_0-\xi)\rho_0'+\rho_0u_0' = 0, \tag{1.19}$$

$$\rho_0(u_0-\xi)u_0'+p_0' = 0, \tag{1.20}$$

$$2p_0/(\gamma-1)+(u_0-\xi)p_0'-(\gamma p_0/\rho_0)\{2\rho_0/(\gamma-1)+(u_0-\xi)\rho_0'\} = 0. \tag{1.21}$$

The solution of these equations near $\xi = 0$, satisfying $u_0(0) = 0$ is found by assuming the expansions

$$\rho_0(\xi) = \xi^{\alpha}(a_0 + a_1\xi^{\sigma} + a_2\xi^{2\sigma} + \ldots), \tag{1.22}$$

$$p_0(\xi) = \xi^{\beta}(b_0 + b_1\xi^{\sigma} + b_2\xi^{2\sigma} + \ldots), \tag{1.23}$$

$$u_0(\xi) = \xi^{\delta}(c_0 + c_1\xi^{\sigma} + c_2\xi^{2\sigma} + \ldots), \tag{1.24}$$

where $\sigma > 0$, $\delta > 0$, $a_0 \neq 0$, $b_0 \neq 0$ and $c_0 \neq 0$. Only one set of values of the indices α, β, δ and σ, leads to a consistent solution. This set is

$$\alpha = \frac{2(\gamma - 1)}{(\gamma^2 - \gamma + 2)}, \quad \beta = 0, \quad \delta = 1 \quad \text{and} \quad \sigma = \frac{2(\gamma^2 + 1)}{(\gamma^2 - \gamma + 2)}.$$

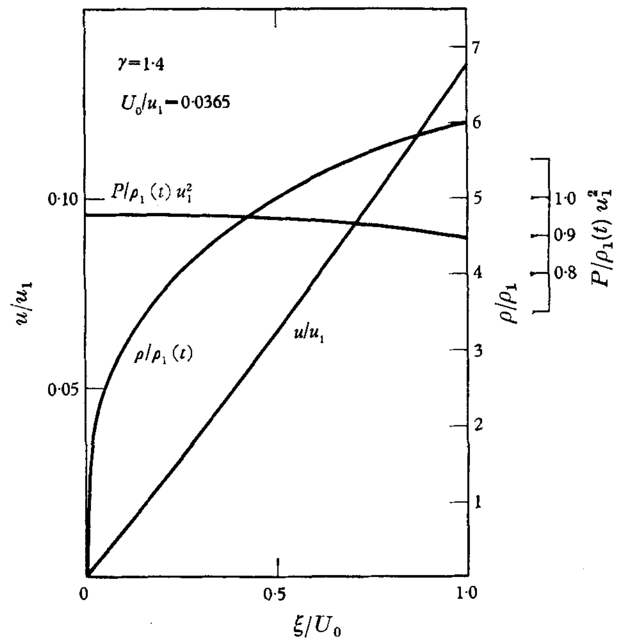

FIGURE 3. Reflexion off a rigid wall—solution. The subscript one denotes values in front of the shock.

a_0 and b_0 may be chosen arbitrarily, but once these are fixed all the remaining coefficients are uniquely determined. A two-parameter family of solutions is thus obtained.

The whole family of solutions can be expressed in terms of one particular solution. Suppose

$$\rho_0(\xi) = f(\xi), \quad p_0(\xi) = g(\xi), \quad u_0(\xi) = h(\xi)$$

is such a particular solution, then it is easily verified that

$$\rho_0(\xi) = Af(\xi/B),$$

$$p_0(\xi) = AB^2 g(\xi/B),$$

$$u_0(\xi) = Bh(\xi/B),$$

where A and B are arbitrary constants, is also a solution satisfying the boundary condition $u_0(0) = 0$. The functions f, g and h are determined by specifying a_0 and b_0 arbitrarily and integrating the ordinary differential equations (1.19) to (1.21) numerically using the series near $\xi = 0$ for starting values. The constants A and B and the initial shock velocity U_0 are then determined by using the shock relations to match the solution for region 2 with the known solution for region 1.

The series (1.14) to (1.18) may be developed further by considering higher-order terms in the equations (1.11) to (1.13) and the shock relations. Alternatively the solution for larger $(t-t_1)$ may be obtained by direct numerical integration of the partial differential equations (1.11) to (1.13), using initial data obtained from the zero-order solution.

The zero-order solution for $\gamma = 1{\cdot}4$ is shown in figure 3 normalized with respect to the values of ρ and u just ahead of the shock. $p_0(\xi)$ is not very sensitive to variations of ξ, confirming Stanyukovich's observation that space-wise pressure variations are small. $u_0(\xi)$ is approximately linear. This means that good approximations to $\rho_0(\xi)$, $p_0(\xi)$ and $u_0(\xi)$ may be obtained by using only the leading terms of the expansions (1.22) to (1.24).

2. Shockwave-front interaction

A weak compressive wave impinging upon the expansion strengthens as it propagates through the region of decreasing density. Ultimately a shock develops somewhere behind the front. The position and time (x_s, t_s) of the onset of the shock formation are easily determined; the calculation is straightforward and many of the details are omitted here.

Let the compression wave be generated by the motion of a piston in the rest gas and let the disturbance reach the expansion at position x_0 and at time $t_0 = -x_0$. It is assumed that the derivatives of u and c are discontinuous across the leading characteristics of this sound wave; the quantities themselves are, of course, continuous. As the wave propagates towards the front, it strengthens and at position (x_s, t_s) the derivatives of u and c become infinite indicating the formation of a shock wave. For compressional waves of this type, position (x_s, t_s) is located on the C_+ characteristic passing through the point (x_0, t_0), denoted by $C_+^{(0)}$, which is

$$x = \frac{2}{\gamma-1}t - \frac{\gamma+1}{\gamma-1}t_0^{2(\gamma-1)/(\gamma+1)}t^{(3-\gamma)/(\gamma+1)}. \tag{2.1}$$

The state of the gas ahead of this wave is known from the fundamental solution, and in particular, along $C_+^{(0)}$,

$$c = \left(\frac{t_0}{t}\right)^{2(\gamma-1)/(\gamma+1)}, \quad u = \frac{2}{\gamma-1}\left[1-\left(\frac{t_0}{t}\right)^{2(\gamma-1)/(\gamma+1)}\right]. \tag{2.2}$$

In terms of the co-ordinate ζ measured with respect to the moving secondary wave front,

$$\zeta = x - t\left[\frac{2}{\gamma-1} - \frac{\gamma+1}{\gamma-1}\left(\frac{t_0}{t}\right)^{2(\gamma-1)/(\gamma+1)}\right] \tag{2.3}$$

($\zeta = 0$ on the curve $C_+^{(0)}$), and the time t, the equations for the Riemann invariants

$$\phi = u + 2c/(\gamma - 1), \tag{2.4}$$

$$\psi = u - 2c/(\gamma - 1), \tag{2.5}$$

become $$\phi_t + \left[\frac{3-\gamma}{\gamma-1}\left(\frac{t_0}{t}\right)^{2(\gamma-1)/(\gamma+1)} - \frac{2}{\gamma-1} + \frac{\gamma+1}{4}\phi + \left(\frac{3-\gamma}{4}\right)\psi\right]\phi_\zeta = 0, \tag{2.6}$$

$$\psi_t + \left[\frac{3-\gamma}{\gamma-1}\left(\frac{t_0}{t}\right)^{2(\gamma-1)/(\gamma+1)} - \frac{2}{\gamma-1} + \frac{3-\gamma}{4}\phi + \frac{\gamma+1}{4}\psi\right]\psi_\zeta = 0. \tag{2.7}$$

If, now, we utilize wave-front expansions of the form

$$\phi = \sum_{n=0}^{\infty} \zeta^n \phi_n(t), \quad \psi = \sum_{n=0}^{\infty} \zeta^n \psi_n(t),$$

it follows that

$$\phi_0 = \frac{2}{\gamma-1}, \quad \psi_0 = \frac{2}{\gamma-1} - \frac{4}{\gamma-1}\left(\frac{t_0}{t}\right)^{2(\gamma-1)/(\gamma+1)}, \tag{2.8}$$

and
$$\left.\begin{aligned} \phi_1 &= \left(\frac{4}{\gamma+1}\right)^2 t^{(3-\gamma)/(\gamma+1)}\left[t^{4/(\gamma+1)} - k\right]^{-1}, \\ \psi_1 &= -\frac{4}{\gamma+1}\frac{1}{t}, \end{aligned}\right\} \tag{2.9}$$

where k is a constant. The last expression shows that the derivative $\phi_1(t) = \phi_\zeta|_{\zeta=0}$ becomes infinite at the time $t_s = k^{\frac{1}{4}(\gamma+1)}$. Since the wave is a compression, let $\phi_1(t_0) = -m > 0$. We can then evaluate the constant k and locate the position of shock formation,

$$\eta_s = x_s/t_s = \frac{2}{\gamma-1} - \frac{\gamma+1}{\gamma-1}\left(\frac{t_0}{t_s}\right)^{2(\gamma-1)/(\gamma+1)}, \tag{2.10}$$

$$t_s/t_0 = \left(1 + \left(\frac{4}{\gamma+1}\right)^2 \Big/ mt_0\right)^{\frac{1}{4}(\gamma+1)}. \tag{2.11}$$

(Note that this position lies behind the gas-vacuum interface.) Additional terms of the series can be computed, as desired, in order to determine the solution behind the compressive wave front, and the formation of shocks in this region. This is not, however, our primary aim.

The shock wave thus created continues to strengthen as it overtakes the gas-vacuum interface. To gain insight into the collision of shock and front, we consider a slightly modified problem in which a strong shock wave impinges upon the expansion. The shock is created by the uniform motion of a piston in the stationary gas. The analysis is based on the characteristic rule, developed by Chisnell (1955), Whitham (1958) and others and is entirely similar to the development contained in Whitham (1958), § 6. Alternatively, the motion of the shock as it nears the gas-vacuum interface can be studied by using a similarity solution. A recent paper by Sakurai (1960) uses this method for a related problem.

The technique or rule is to apply the relationship

$$dP + \rho c\, du = 0, \tag{2.12}$$

which is valid along a C_+ characteristic, along the shock front. The flow behind the shock is non-homentropic and the strong shock relationships are

$$u = \frac{2}{\gamma+1}U, \quad \rho = \frac{\gamma+1}{\gamma-1}\rho_i, \quad P = \frac{2}{\gamma+1}\rho_i U^2. \tag{2.13}$$

Here, U is the dimensionless shock velocity ($c_0 U$ is the dimensional quantity) and $\rho_i = \rho_i(\eta)$ is the density in front of the shock wave. Substitution of these formulas into equation (2.12) results in a differential expression relating the shock velocity to the density

$$d\rho/\rho + \left\{2 + \left(\frac{2\gamma}{\gamma-1}\right)^{\frac{1}{2}}\right\} dU/U = 0. \tag{2.14}$$

Therefore, U can be obtained as a function of ρ by a quadrature. Since the density is itself solely a function of the variable $\eta = x/t$, the velocity is too and, in fact,

$$U = U(\eta) = U(-1)\rho^{-\Omega}, \tag{2.15}$$

where

$$\Omega = 2 + \left(\frac{2\gamma}{\gamma-1}\right)^{\frac{1}{2}}. \tag{2.16}$$

U becomes infinite as the shock approaches the gas-vacuum interface and $\rho(\eta)$ nears zero. It follows from (2.13) that the particle velocity also increases indefinitely. At the impact of the shock upon the interface, the particles constituting the front instantaneously accelerate to infinite velocity. Since the sound speed is non-negative, the Lagrangian equation governing the motion of the frontal particles

$$u_t = -\frac{2}{\gamma-1}\lim_{c\to 0} cc_x$$

shows that the acceleration is always non-negative too, i.e. the frontal particles can never decelerate. We conclude that the interface is instantaneously projected to infinity by an impinging shock wave.

Thus the motion of the front is unstable with respect to small compressive disturbances forming in the interior of the flow. For this reason, we shall subsequently confine our attention to a study of gas expansions in the initial stages of motion at small times.

3. Spherical and cylindrical fronts

A uniform stationary gas occupying the region interior or exterior to a sphere (or cylinder) of unit dimensionless radius is allowed to expand freely into the complementary vacuous space. In the initial stages of the motion, a sound wave propagates into the quiescent gas as the gas-vacuum interface proceeds into vacuum. The Eulerian equations governing such symmetric flows are

$$u_t + uu_r + 2(\gamma-1)^{-1}cc_r = 0, \tag{3.1}$$

$$2(\gamma-1)^{-1}(c_t + uc_r) + cu_r + \sigma cu/r = 0, \tag{3.2}$$

where $\sigma = 0, 1, 2$ corresponds to the one-, two- or three-dimensional problems. Alternatively, these equations can be written in terms of the characteristic variables, as

$$u_\xi + \frac{2}{\gamma-1} c_\xi + \sigma \frac{uc}{r} t_\xi = 0, \quad r_\xi = (u+c)\,t_\xi; \tag{3.3}$$

$$u_\eta - \frac{2}{\gamma-1} c_\eta + \sigma \frac{uc}{r} t_\eta = 0, \quad r_\eta = (u-c)\,t_\eta. \tag{3.4}$$

The boundary conditions require only that $u = 0$ on the curve $c = 1$, the leading characteristic of the sound wave propagating into the stationary gas. The gas-vacuum interface is located by the requirement $c = 0$.

The objective here is to show that the gas-vacuum front propagates with the constant velocity $2/(\gamma-1)$ for all symmetric motions of this type just as it does in the one-dimensional problem. The frontal motion is essentially unaffected by the geometrical factor $\sigma cu/r$ appearing in (3.2). This is perhaps not so surprising in the case of divergent motion (by which we shall mean that the gas is initially contained within a sphere or cylinder) for then the extra term plays the role of a retarding force on the motion of the front. But it is easily shown that the front cannot decelerate so that the magnitude of this force at the interface must be zero. Indeed we may argue that the front is either a characteristic or an envelope of such curves. Consider the latter case, and let $\eta(r, t) =$ constant be a C_+ characteristic which emanates from the stationary gas and touches the interface. Upon integrating equation (3.3) along the entire length of this characteristic from ξ_0 to ξ_f (the front), we find that

$$u_{(f)} = \frac{2}{\gamma-1} - \int_{\xi_0}^{\xi_f} \sigma \frac{uc}{r} t_\xi \, d\xi, \tag{3.5}$$

where the subscript f denotes values at the interface. For small t at least, the quantity uc/r is certainly non-negative along the entire characteristic, and this implies that the integral appearing in (3.5) is positive since $t_\xi d\xi = dt > 0$. It follows then, that

$$u_{(f)} \leqslant \frac{2}{\gamma-1}.$$

However, the front must begin to move with the velocity appropriate to the local one-dimensional configuration, that is $u = 2/(\gamma-1)$, and if the inequality sign were to hold in the preceding expression, we could only conclude that the front must, at some time, decelerate. That this is impossible is a direct consequence of the Lagrangian equation governing the motion of the frontal particle,

$$\frac{\partial u_{(f)}}{\partial t} = -\frac{2}{\gamma-1}\left(c\frac{\partial c}{\partial r}\right)_{(f)}. \tag{3.6}$$

The sound speed is a non-negative quantity which increases away from the front, $c = 0$, i.e.

$$(cc_r)_{(f)} \leqslant 0.$$

The acceleration of the frontal particle is then non-negative, showing that it cannot decelerate at any time. Therefore, the front must be a characteristic curve (not an envelope) and the velocity of the diverging interface is a constant

$$u_{(f)} = \frac{2}{\gamma-1}. \tag{3.7}$$

(If a co-ordinate $\zeta(r,t)$ orthogonal to η is introduced, there results from equation (3.3) the relationship
$$u_{(f)} + \frac{2}{\gamma-1}c_{(f)} = \frac{2}{\gamma-1},$$
which is equation (3.7) since $c_{(f)} \equiv 0$.) Consequently, no sound waves (characteristics) from the interior reach the front in the initial stages. Compressional disturbances may arise at later times due to instabilities and these develop into shock waves, as discussed in § 2, which intersect the front.

The converging front presents a more subtle case, and the preceding arguments do not apply. The geometrical factor now tends to accelerate the front if it has any influence at all. That it indeed exerts no effect on the front whatsoever is suggested by the belief that the forcing term to the right in equation (3.6) is probably zero at small times, since it is so in the one-dimensional problem. Only when (if) the gradient, c_r, becomes infinite so that
$$\lim_{c\to 0} cc_r \neq 0$$
can the front accelerate and this necessitates the formation of an envelope. It is difficult to show conclusively that envelopes do not form in the initial stages, and it is best to postpone a continued discussion of this point pending further analytical investigation.

We take as basic dependent variables the one-dimensional Riemann invariants ϕ, ψ (equations (2.4) and (2.5)). In terms of a 'similarity' co-ordinate system consisting of the time t and the variable
$$\eta = (r-1)/t, \tag{3.8}$$
equations (3.1) and (3.2) become
$$(1+\eta t)\,[t\phi_t - \eta\phi_\eta + \tfrac{1}{4}\{(\gamma+1)\,\phi + (3-\gamma)\,\psi\}\,\phi_\eta] + \tfrac{1}{8}\sigma(\gamma-1)\,t(\phi^2-\psi^2) = 0, \tag{3.9}$$
$$(1+\eta t)\,[t\psi_t - \eta\psi_\eta + \tfrac{1}{4}\{(3-\gamma)\,\phi + (\gamma+1)\,\psi\}\,\psi_\eta] - \tfrac{1}{8}\sigma(\gamma-1)\,t(\phi^2-\psi^2) = 0. \tag{3.10}$$
(If the first of these is represented symbolically as $L(\phi,\psi) = 0$, the second is $L(\psi,\phi) = 0$.) The boundary conditions require that $u = 0, c = 1$ on the leading characteristic of the sound wave propagating into the stationary gas. That is to say $\phi = -\psi = 2/(\gamma-1)$ on $\eta = -1$ $(r = 1-t)$ if the gas-vacuum front is diverging, or on $\eta = 1$ $(r = 1+t)$ in the case of convergence. Again, the condition $c = 0$ locates the interface.

For future use, note that if $\phi(\eta,t)$ and $\psi(\eta,t)$ are solutions of the differential equations (3.9) and (3.10), then
$$\phi^*(\eta,t) = -\psi(-\eta,\,-t), \quad \psi^*(\eta,t) = -\phi(-\eta,\,-t) \tag{3.11}$$
are also solutions of the same equations.

Consider first the solution of the boundary-value problem for which the gas-vacuum interface is a diverging front. We restrict our attention to $t \leqslant 1$, that is before the sound wave reaches the symmetry axis. We seek a solution of the form
$$\phi = \sum_{n=0}^{\infty} A_n(\eta)\,t^n, \quad \psi = \sum_{n=0}^{\infty} B_n(\eta)\,t^n, \tag{3.12}$$

satisfying the foregoing boundary conditions. Once these coefficients are computed, the solution of the converging front problem denoted by asterisks is given by equation (3.11)

$$\phi^* = -\sum_0^\infty B_n(-\eta)(-1)^n t^n, \quad \psi^* = -\sum_0^\infty A_n(-\eta)(-1)^n t^n,$$

with $\phi^* = -\psi^* = 2/(\gamma-1)$ on $\eta = 1$ and $\eta \leqslant 1$; $0 \leqslant t$. It follows from these expressions that the radii of convergence of both series representing diverging and converging expansions are identical. Consequently, we need only consider the case of divergent motion to completely solve both problems.

The replacement of the series of (3.12) into equations (3.9) and (3.10) leads to a set of non-linear ordinary differential equations for the coefficient functions $A_n(\eta)$, $B_n(\eta)$ which are, for $n = 0$,

$$\left.\begin{aligned}(-\eta+\tfrac{1}{4}(\gamma+1)A_0+\tfrac{1}{4}(3-\gamma)B_0)\,dA_0/d\eta &= 0,\\ (-\eta+\tfrac{1}{4}(3-\gamma)A_0+\tfrac{1}{4}(\gamma+1)B_0)\,dB_0/d\eta &= 0,\end{aligned}\right\} \tag{3.13}$$

and for $n \geqslant 1$

$$S_n(\{A_i\},\{B_i\})+\eta S_{n-1}(\{A_i\},\{B_i\}) = -T_{n-1}(\{A_i\},\{B_i\}) \tag{3.14}$$

and

$$S_n(\{B_i\},\{A_i\})+\eta S_{n-1}(\{B_i\},\{A_i\}) = T_{n-1}(\{B_i\},\{A_i\}), \tag{3.15}$$

where

$$S_n(\{A_i\},\{B_i\}) = nA_n-\eta\frac{dA_n}{d\eta}+\frac{1}{4}\sum_{k=0}^{n}((\gamma+1)A_{n-k}+(3-\gamma)B_{n-k})\frac{dA_k}{d\eta} \tag{3.16}$$

and

$$T_n(\{A_i\},\{B_i\}) = \tfrac{1}{8}\sigma(\gamma-1)\sum_{k=0}^{n}(A_{n-k}A_k-B_{n-k}B_k). \tag{3.17}$$

(For example $\quad T_0(\{A_i\},\{B_i\}) = \tfrac{1}{8}\sigma(\gamma-1)(A_0^2-B_0^2)$

and $\quad T_n(\{A_i\},\{B_i\}) = -T_n(\{B_i\},\{A_i\})$.)

The boundary conditions require that

$$A_n(-1) = -B_n(-1) = \frac{2}{\gamma-1}\delta_{0n}. \tag{3.18}$$

The appropriate solution of equation (3.13) is

$$A_0 = \frac{2}{\gamma-1}, \quad B_0 = \frac{4}{\gamma+1}\left(\eta-\frac{1}{2}\frac{3-\gamma}{\gamma-1}\right), \tag{3.19}$$

and these zeroth-order terms are, in fact, the complete solution of the one-dimensional problem, $\sigma = 0$ (see equations (1.3) and (1.4)).

It is convenient to introduce the variable

$$z = \frac{\gamma-1}{\gamma+1}\left(\frac{2}{\gamma-1}-\eta\right) \tag{3.20}$$

and henceforth to consider all coefficients and dependent functions to be functions of z and t (instead of η and t), i.e.

$$A_n = A_n(z), \quad \phi = \phi(z,t), \quad \text{etc.}$$

Accordingly,

$$A_0 = \frac{2}{\gamma-1}, \quad B_0 = \frac{2}{\gamma-1}(1-2z), \tag{3.21}$$

and the next two coefficients are then

$$\left.\begin{aligned} A_1(z) &= -\frac{4\sigma}{\gamma+1}\frac{\omega z^{\omega}}{(2-\omega)(1-\omega)}+\frac{4\sigma}{\gamma+1}\omega^2\left[\frac{z}{1-\omega}-\frac{z^2}{2-\omega}\right],\\ B_1(z) &= \frac{2\sigma\omega z(1-z)}{\gamma+1}-\frac{2\sigma(3-\gamma)\,\omega^2}{(\gamma+1)^2}\left[\frac{z}{1-\omega}-\frac{z^2}{2-\omega}-\frac{z^{\omega}}{(2-\omega)(1-\omega)}\right], \end{aligned}\right\} \tag{3.22}$$

where $2\omega = (\gamma+1)/(\gamma-1)$. In particular, $A_1 = B_1 = 0$ at $z = 0$ $(\eta = 2/(\gamma-1))$ and it can be shown that this is a property common to all subsequent coefficients, i.e.

$$A_n = B_n = 0 \quad \text{for} \quad z = 0, \quad n \geqslant 1.$$

In order to prove this assertion, equations (3.14) and (3.15) are first manipulated into the form

$$z\frac{dA_n}{dz}-\frac{n}{2}\left(\frac{\gamma+1}{\gamma-1}\right)A_n = -\frac{1}{\gamma}\sum_{k=1}^{n-1}[(\gamma+1)A_{n-k}+(3-\gamma)B_{n-k}]\frac{dA_k}{dz}$$
$$+\frac{1}{2}\frac{\gamma+1}{\gamma-1}\sum_{k=1}^{n}\left(\frac{2}{\gamma-1}\right)^{k-1}\left(\frac{\gamma+1}{2}z-1\right)^{k-1}T_{n-k}(\{A_i\},\{B_i\}), \tag{3.23}$$

$$(n+1)B_n+\frac{3-\gamma}{\gamma+1}A_n = \frac{1}{4}\frac{\gamma-1}{\gamma+1}\sum_{k=1}^{n-1}[(\gamma+1)B_{n-k}+(3-\gamma)A_{n-k}]\frac{dB_k}{dz}$$
$$+\sum_{k=1}^{n}\left(\frac{2}{\gamma-1}\right)^{k-1}\left(\frac{\gamma+1}{2}z-1\right)^{k-1}T_{n-k}(\{A_i\},\{B_i\}). \tag{3.24}$$

(The equations for the coefficients A_1, B_1, are of the same form.)

Let $\mathscr{R}$ denote the class of functions consisting of a linear finite sum of terms of the form $Kz^{\nu}\ln^m z$, where $\nu \geqslant 1$, m is a positive integer and K is a constant. If $p(z)$ is a typical element of this set, then

$$p(z) = z^{\nu_0}[K_0^{(0)}+K_1^{(0)}\ln z+\ldots+K_{m_0}^{(0)}\ln^{m_0}z]+\ldots$$
$$+z^{\nu_j}[K_0^{(\gamma)}+K_1^{(\gamma)}\ln z+\ldots+K_{m_j}^{(\gamma)}\ln^{m_j}z].$$

We prove first that all the coefficient functions A_n, B_n are members of $\mathscr{R}$. An induction argument is used; it is shown that if the functions $A_1 \ldots A_{n-1}$, $B_1 \ldots B_{n-1}$ belong to class $\mathscr{R}$, then so do A_n and B_n. The functions A_1 and B_1 certainly satisfy this condition. Furthermore, it is easily verified that such combinations as, for example,

$$\{(\gamma+1)B_{n-k}+(3-\gamma)A_{n-k}\}dB_k/dz, \quad T_k(\{A_i\},\{B_i\}), \quad \text{etc.},$$

are members of $\mathscr{R}$ for $0 \leqslant k \leqslant n-1$. Therefore, equation (3.23) is of the form

$$z\frac{dA_n}{dz}-\frac{n}{2}\frac{\gamma+1}{\gamma-1}A_n = p(z) \tag{3.25}$$

which has the homogeneous solution

$$kz^{\lambda}$$

with $\lambda = \frac{1}{2}n(\gamma+1)/(\gamma-1) \geqslant 1$. The particular solution corresponding to an inhomogeneous term of the form

$$z^{\nu}\ln^m z$$

is

$$z^{\nu}\sum_{k=0}^{m}\frac{(-1)^k m!}{(m-k)!}\frac{\ln^{m-k}z}{(\nu-\lambda)^{k+1}}$$

if $\nu \neq \lambda$ and $z^{\nu} \ln^{m+1} z/(m+1)$ in the exceptional circumstance $\nu = \lambda$. In either case, the particular solution satisfying the boundary condition belongs to class $\mathscr{R}$, i.e. $A_n(z)$ belongs to $\mathscr{R}$, $n \geqslant 1$. Furthermore, equation (3.24) implies that $B_n(z)$ belongs to $\mathscr{R}$, and this completes the proof. It should be noted that the boundary condition $B_n(1) \equiv 0$ is automatically satisfied.

Every function $p(z)$ belonging to $\mathscr{R}$ has the following properties:

$$\text{(i)} \quad p(0) = 0,$$

$$\text{(ii)} \quad \left|\frac{dp}{dz}\right| < \infty \quad \text{for all finite values of } z.$$

Since $A_0(0) = -B_0(0) = 2/(\gamma - 1)$ it follows directly that

$$c = 0,$$

$$u = \frac{2}{\gamma - 1} \quad \text{on the curve } z = 0.$$

In other words, the gas-vacuum interface is the curve $z = 0$ $(\eta = 2/(\gamma-1))$ and this front moves with constant velocity.

Although we considered only the case divergent of motion, equation (3.11) *et seq.* imply that the proof is also valid for convergent motion as well. For such flows, the variable η is to be replaced by $-\eta$, in all calculations, so that, for example, $z = (\gamma-1)/(\gamma+1)\{2/(\gamma-1)+\eta\}$, etc. The analysis then proceeds in exactly the same way. The notation for both cases is summarized in Table 1.

The gas-vacuum interface moves with constant velocity unaffected by geometrical configuration, in both the convergent or divergent expansions.

The specific coefficient functions, for the values $\gamma = \frac{5}{3}$, $\sigma = 2$, are illustrated in figure 4. The first few coefficients are

$$\left.\begin{aligned} &A_0 = 3, \quad A_1 = -12(z - z^2 + z^2 \ln z), \\ &A_2 = 36[z - 2z^2 + \tfrac{8}{3}z^3 + \tfrac{1}{3}z^4 + (z^2 \ln z)\{2 - \tfrac{7}{3}z + \tfrac{7}{3}z \ln z\}], \\ &B_0 = 3(1-2z), \quad B_1 = 6(z - z^2 + \tfrac{1}{2}z^2 \ln z), \\ &B_2 = -12(\tfrac{3}{2}z - 3z^2 + \tfrac{4}{3}z^3 + \tfrac{1}{6}z^4) - (z^2 \ln z)(18 - \tfrac{35}{2}z + 15z \ln z). \end{aligned}\right\} \tag{3.26}$$

The velocity and sound-speed coefficients, u_n and c_n, are shown in figure 5. The deviation of c from the one-dimensional value is entirely negative (positive) for divergent (convergent) motion. The corresponding deviation of the velocity u changes sign somewhere in the expansion. The absolute magnitude of u increases (decreases) behind the front for convergent (divergent) flows as a result of the focusing due to the geometry.

Stanyukovich (1960) in dealing with the divergent spherical problem takes the Riemann invariant ϕ to be a constant throughout the expansion. The solution thus obtained ($\gamma = 3$ only) shows no oscillation of the velocity profile about the one-dimensional case and indeed does not satisfy the energy equation. Although the procedure calls for continued iterations, these are difficult to carry out.

The question of the convergence of the series expansions for ϕ and ψ remains. Obviously we would like to prove conclusively that the series converge uniformly for appropriately small values of t at any position in the rarefaction

$0 \leqslant z \leqslant 1$. This would eliminate the possibility that shock waves form in the interior of the flow at small times as they do in the case of the expansion of a gas in a spherical container into another gas at rest (Friedman 1961). (In such motions, the characteristics reflected from the shock front are responsible for the interior shock wave; here no such phenomena occur.) In view of the complexity of the recurrence formulas interrelating the coefficient functions, such a task is indeed formidable. Some progress has been made in this direction and there is available sufficient evidence to make this assertion convincing, although a complete proof is, as yet, lacking. For these reasons, we present and summarize the evidence here and defer the details of this phase of the investigation to a future report.

Quantity	Divergent motion	Convergent motion
$\eta =$	$\dfrac{r-1}{t}$	$\dfrac{r-1}{t}$
$z =$	$\dfrac{\gamma-1}{\gamma+1}\left(\dfrac{2}{\gamma-1}-\eta\right)$	$\dfrac{\gamma-1}{\gamma+1}\left(\dfrac{2}{\gamma-1}+\eta\right)$
$\phi =$	$\sum_0^\infty A_n(z)\, t^n$	$-\sum_0^\infty (-1)^n B_n(z)\, t^n$
$\psi =$	$\sum_0^\infty B_n(z)\, t^n$	$-\sum_0^\infty (-1)^n A_n(z)\, t^n$
$A_n(z)$, $B_n(z)$ satisfy	equations (3.23), (3.24)	equations (3.23), (3.24)
$u = \frac{1}{2}(\phi+\psi)$	$\sum_0^\infty u_n(z)\, t^n$	$-\sum_0^\infty (-1)^n u_n(z)\, t^n$
$c = \frac{1}{4}(\gamma-1)(\phi-\psi)$	$\sum_0^\infty c_n(z)\, t^n$	$\sum_0^\infty (-1)^n c_n(z)\, t^n$
$u_n(z) =$	$\frac{1}{2}(A_n+B_n)$	$\frac{1}{2}(A_n+B_n)$
$c_n(z) =$	$\frac{1}{2}(\gamma-1)(A_n-B_n)$	$\frac{1}{2}(\gamma-1)(A_n-B_n)$
Gas-vacuum interface, $z=0$	$\eta = 2/(\gamma-1)$	$\eta = -2/(\gamma-1)$
Sound front,	$\eta = -1$	$\eta = 1$
Gas motion in the interval $0 \leqslant z \leqslant 1$	$0 \leqslant t$	$0 \leqslant t$

TABLE 1

It would appear from the computed functions that the particular expansions $\sum_0^\infty A'_n(0)\, t^n$, $\sum_0^\infty B'_n(0)\, t^n$ probably represent extreme cases in matters of convergence, for these are the most highly differentiated terms evaluated at the most crucial physical position. These series can be shown to converge at least for values $\gamma \leqslant \frac{5}{3}$ with radii of convergence exactly equal to $t = 2/(\gamma-1)$. This is, in fact, the time required for the gas-vacuum interface to reach the centre in the case of convergent motion and represents the smallest characteristic time occurring in the entire phenomenon. Furthermore, for $\gamma = \frac{5}{3}$, the expansions are even summable and it is found that in the case of divergent motion

$$\phi_z(0,t) = -\frac{12t}{1+3t}, \quad \psi_z(0,t) = -6\left(\frac{1+2t}{1+3t}\right), \tag{3.27}$$

whereas for convergent flows

$$\phi_z(0,t) = \frac{6(1-2t)}{1-3t}, \quad \psi_z(0,t) = -\frac{12t}{1-3t}. \tag{3.28}$$

These simple expressions can also be independently determined directly from the equations of motion by a 'wave-front' analysis similar to that of § 2. Let

$$g = \phi_z(0,t), \quad f = \psi_z(0,t).$$

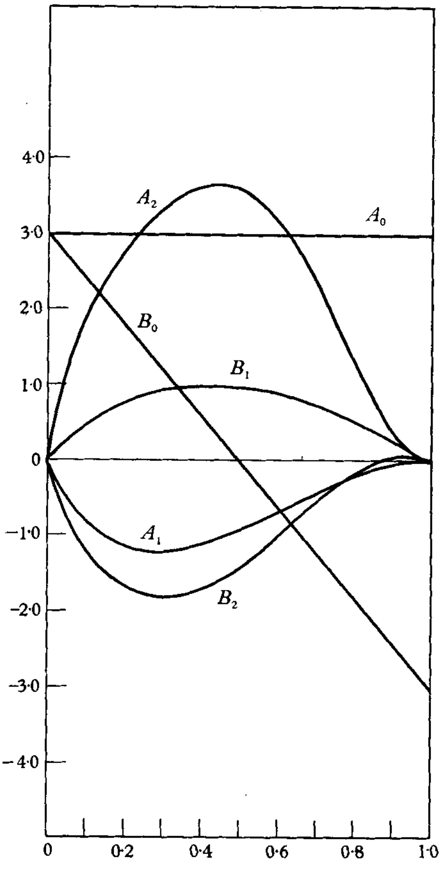

FIGURE 4. Coefficient functions for spherical flow problems, $\gamma = \frac{5}{3}$.

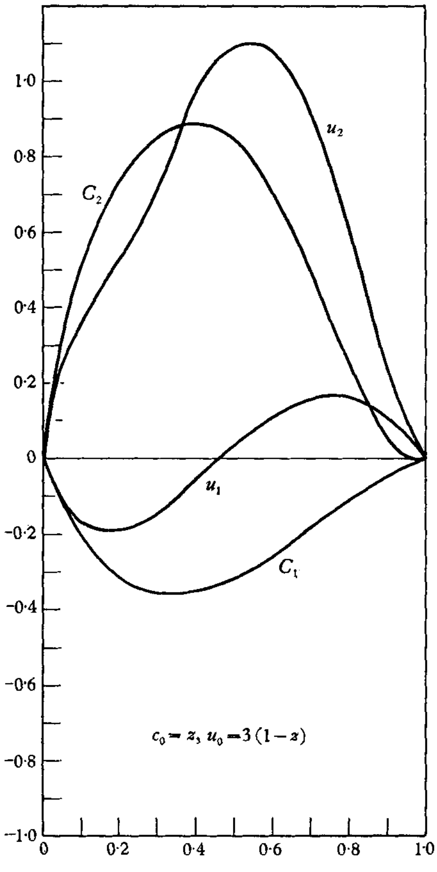

FIGURE 5. Coefficient functions for spherical flow problems, $\gamma = \frac{5}{3}$.

If equations (3.9) and (3.10) (suitably rewritten) are differentiated with respect to z, and the limit taken as z becomes zero, two equations for f and g result,

$$tg' - \left(\frac{\gamma-1}{\gamma+1}\right)\left(\frac{\gamma+1}{4}g + \frac{3-\gamma}{4}f + \frac{\gamma+1}{\gamma-1}\right)g + \frac{t}{1+2t(\gamma-1)}(g-f) = 0,$$

$$tf' - \left(\frac{\gamma-1}{\gamma+1}\right)\left(\frac{3-\gamma}{4}g + \frac{\gamma+1}{4}f + \frac{\gamma+1}{\gamma-1}\right)f - \frac{t}{1+2t(\gamma-1)}(g-f) = 0.$$

The foregoing formulas contained in (3.27) and (3.28) ($\gamma = \frac{5}{3}$) are solutions of these equations. No other closed-form solutions have been found.

Another wave-front analysis in the neighbourhood of the sound wave, $z = 1$, shows that here too, the series are both convergent and summable for all γ. The results are

$$\phi_z(1,t) = 0, \quad \psi_z(1,t) = \frac{4}{\gamma-1}\frac{t}{1-t}\frac{1}{\ln(1-t)} \tag{3.29}$$

for divergent motion and

$$\phi_z(1,t) = \frac{4}{\gamma-1}\frac{t}{(1+t)\ln(1+t)}, \quad \psi_z(1,t) = 0 \tag{3.30}$$

for convergent motion. Thus, the particular series expansions converge at either side of the rarefaction and it seems plausible then to assume convergence throughout the entire interval $0 \leqslant z \leqslant 1$. The value of the derivatives on either side of the flow régime show that the velocity profile does indeed oscillate about the one-dimensional distribution as previously discussed.

No attempt has thus far been made to examine convergence for values of $\gamma > \frac{5}{3}$ although this is a most interesting question. Various similarity solutions have been obtained (to be published in a forthcoming report by C. Hunter) for which the collapsing front moves with constant velocity even near the centre if γ is small enough. On the other hand, for values of γ which are large, in particular $\gamma = 7$, it is known (Hunter 1960) that the interface accelerates. There is evidently a critical value of γ which separates the two régimes. The exact nature and behaviour of the flow as a function of γ is a topic currently under investigation.

The solution of the collisionless Boltzmann equation for spherical expansions does not give significantly different results at early times from the corresponding one-dimensional analysis.

4. Conclusion

The expansion of a gas into vacuum from uniform rest conditions is characterized by an essential redistribution of available energy so that unsteady escape velocity is significantly greater than the steady counterpart. The analytical investigation indicates that a particle in the gas-vacuum interface moves with a uniform escape velocity of dimensionless magnitude $2/(\gamma-1)$ in a direction perpendicular to the original surface after an instantaneous acceleration. The interface apparently moves independently of the remainder of the gas until such time when either the frontal particles collide (which would occur if the original container were somewhere concave) or until dispensed with by a shock wave formed on the interior. A square container, figure 6, would expand as indicated. Each corner opens into a quarter circle and the entire interface eventually becomes circular. The mass, energy, etc., are, however, concentrated in four jet-like structures each moving perpendicular to a face of the original figure.

The study of a collapsing spherical cavity shows that geometry does not alter these statements, at least in the early stages of motion. There exist infinitely many similarity solutions for moderate values of γ which are valid for motion near the centre and have the property that the front moves with constant velocity. The manner in which the short-time solution connects with a particular similarity solution has not been studied as yet.

The value of γ seems to be a critical factor since if this parameter is large enough, the front actually accelerates, i.e. becomes an envelope of characteristics. The critical number dividing the two domains for which the front is either an envelope of characteristics (large γ) or a genuine characteristic itself (smaller γ) is presently under investigation.

The condition that the gas be initially at rest and of uniform pressure and density is also essential. If, for example, the pressure density or sound speed initially increased away from the container wall, the resultant interface accelerates after release (Keller 1957). The front is then an envelope. On the other hand, if the foregoing quantities decrease or remain constant from the container, the interface will move with constant velocity.

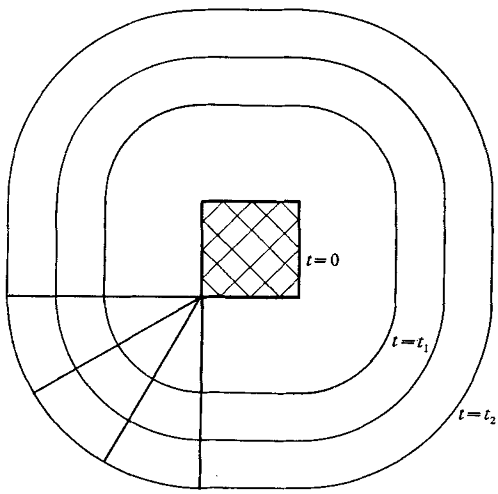

FIGURE 6. Expansion of a square.

From the series expansions we find that the one-dimensional solution is a fairly good approximation to the spherical flow pattern for a moderate length of time, of the order of one-third the total time for the front to reach the centre.

In a sense, the gas-vacuum interface can conceivably serve to locate approximately the position of a strong shock which would develop if the expansion took place into a uniform quiescent gas of lower density. The simplicity with which this determination can be made suggests the possibility of an approximate iterative theory based on the related frontal motion as a first step.

For example, in this speculative vein, consider the efflux of one rest gas in a pipe into a stationary gaseous medium. If the density of the external medium were much smaller than the density of the gas in the pipe, we would be tempted to consider the corresponding gas-vacuum flow, associating the position of the interface with the location of the strong unsteady shock as a first approximation. The particles on the original surface all translate to the right (figure 7) with escape speed. Particles on the pipe wall execute an instantaneous Prandtl–Meyer expansion from a known state into vacuum at the open end and then proceed into vacuum in a direction and with the velocity thus obtained. In this manner, a continually developing surface is achieved which evolves into the steady-state configuration. The upper portion of the curved frontal surface is given parametrically in terms of polar co-ordinates (r, θ), measured from the pipe edge, as

$$r = u_w(t-\tau),$$

$$\theta = \frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{1}{2}}\left(\frac{3\pi}{2}-\sin^{-1}\left\{\frac{2\gamma-M^2(\gamma-1)}{2+(\gamma-1)M^2}\right\}\right)+\tfrac{1}{4}\pi-\tfrac{1}{2}\sin^{-1}\frac{M^2-2}{M^2},$$

where

$$u_w = \left(u_i^2+\frac{2}{\gamma-1}c_i^2\right)^{\frac{1}{2}},$$

$$u_i = \frac{2}{\gamma-1}\left(1+\frac{1}{\tau}\right), \quad c_i = \frac{\gamma-1}{\gamma+1}\left(\frac{2}{\gamma-1}-\frac{1}{\tau}\right)$$

and

$$t \geqslant \tau \geqslant \tfrac{1}{2}(\gamma-1), \quad M = u_i/c_i.$$

(The pattern is symmetric about the axis.) Further effort is required to investigate this approach.

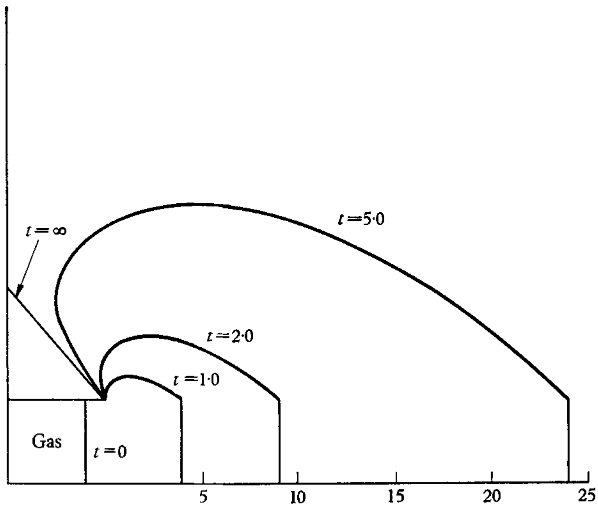

FIGURE 7. Expansion from a tube into vacuum.

This research, initiated at the Los Alamos Scientific Laboratory, was partially supported by the U.S. Air Force under Contract AF 49 (638)–708, monitored by the Air Force Office of Scientific Research of the Air Research and Development Command.

REFERENCES

CHISNELL, R. F. 1955 *J. Fluid Mech.* **2**, 286.

COPSON, E. T. 1950 *Mon. Not. R. Ast. Soc.* **110**, 238.

FRIEDMAN, M. F. 1961 *J. Fluid Mech.* **11**, 1.

HUNTER, C. 1960 *J. Fluid Mech.* **8**, 241.

KELLER, J. B. 1948 *Comm. Pure Appl. Math.* **1**, 275.

KELLER, J. B. 1957 *Quart. Appl. Math.* **14**, 171–184.

PACK, D. C. 1953 *Mon. Not. R. Ast. Soc.* **113**, 43–51.

SAKURAI, A. 1960 *Comm. Pure Appl. Math.* **13**, 353.

STANYUKOVICH, K. P. 1960 *Unsteady Motion of Continuous Media.* New York: Pergamon Press.

WHITHAM, G. B. 1958 *J. Fluid Mech.* **4**, 337.